

TRACES AND CHARACTERISTIC CLASSES IN INFINITE DIMENSIONS

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1. INTRODUCTION

Note: Parts of §2.3 are not correct. The equivariant curvatures $\tilde{\Omega}_u$ and $\tilde{\Omega}_u^E$ introduced below (2.4) take values in first order differential operators, by the work of T. McCauley [15]. Thus the leading order trace of powers of these operators is not defined. In particular, we do not have a definition of the leading order equivariant Chern character or \hat{A} -class. Thus Theorem 2.4(ii) and Theorem 2.5 are not correct.

Infinite rank vector bundles often appear in mathematics and mathematical physics. As one example, tangent bundles to spaces of maps $\text{Maps}(N, M)$ from one manifold to another are important in string theory and in formal proofs of the Atiyah-Singer theorem on loop spaces. In addition, gauge theories use bundles associated to the basic fibration $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{G}$ of connections to connections modulo gauge transformations. Usually one focuses on finite dimensional associated moduli spaces to produce Gromov-Witten invariants and Donaldson/Seiberg-Witten invariants. In this paper, we discuss the construction of characteristic classes directly on these infinite rank bundles and their applications to topology. The main results are the construction of a universal \hat{A} -polynomial and Chern character that control the S^1 -index theorem for all circle actions on a fixed vector bundle over a manifold (Thms. 2.4, 2.5), and the detection of elements of infinite order in the diffeomorphism groups of 5-manifolds associated to projective algebraic Kähler surfaces (Thms. 3.6, 3.7).

These characteristic classes are modeled on Chern classes and Chern-Simons classes for complex vector bundles, but with the structure group $U(n)$ replaced by a gauge group $\mathcal{G} = \text{Aut}(E)$ or a larger group ΨDO_0^* of zeroth order invertible pseudodifferential operators (ΨDO s) acting on sections of a bundle E over a closed manifold M . Since finite rank Chern classes depend essentially on the ordinary matrix trace on $\mathfrak{u}(n)$, it is natural to look for traces on the Lie algebra of ΨDO_0^* . These traces come in two types: one is built from the leading order symbol of a ΨDO , and in the gauge group case is just $\int_M \text{tr}(A) \, \text{dvol}$ for $A \in \Gamma(\text{End}(E)) = \text{Lie}(\mathcal{G})$. The second is built from the Wodzicki residue of a ΨDO . These traces are quite different, in that the Wodzicki trace vanishes on $\Gamma(\text{End}(E))$, but they share the crucial locality property that they are both integrals

of pointwise computed functions. Thus characteristic classes built from these traces are in theory as computable as finite rank Chern classes, a distinct advantage over the usual operator trace.

The leading order trace is fairly easy to work with. For example, the tangent bundle to the loop space LM is the sheaf-theoretic pushdown of ev^*TM for the evaluation map $\text{ev} : LM \times S^1 \rightarrow M$, and the leading order Pontrjagin classes of TLM are related to the Pontrjagin classes of M . In particular, these leading order classes are often nonzero. We can use these classes to restate the S^1 -index theorem as a statement on LM and to construct an equivariant universal \hat{A} -polynomial on $LM \times \mathcal{B}$, with \mathcal{B} the space of metrics on M , which appears in the S^1 -index theorem for every action on M . We extend this to twisted Dirac operators by constructing a universal Chern character.

In contrast, the Wodzicki version of characteristic classes seems to be unrelated to the finite dimensional theory. While the Wodzicki-Pontrjagin classes vanish for TLM and conjecturally on all ΨDO_0^* -bundles, the associated secondary/Chern-Simons classes are sometimes nonzero. These WCS classes on TLM can detect nontrivial elements in $\pi_1(\text{Diff}(\overline{M}_k))$ for many Sasakian 5-manifolds \overline{M}_k , $k \in \mathbb{Z} \setminus \{0\}$. These manifolds are the total spaces of circle bundles over projective algebraic Kähler surfaces M , and come in infinite families for each such M .

In §2, we discuss leading order classes, and in §3 we discuss the Wodzicki classes. One common theme is the use of S^1 actions $a : S^1 \times M \rightarrow M$ on compact manifolds. Any action gives rise to both a map $a^L : M \rightarrow LM$, $a^L(m)(\theta) = a(m, \theta)$, taking a point to its orbit, and a map $a^D : S^1 \rightarrow \text{Diff}(M)$ given by $a^D(\theta)(m) = a(\theta, m)$. This is just the set theory equality $\text{Maps}(X \times Y, Z) = \text{Maps}(X, \text{Maps}(Y, Z)) = \text{Maps}(Y, \text{Maps}(X, Z))$ for $X = S^1, Y = Z = M$.

We use a^L in §2 to discuss the S^1 -index theorem. To state the main result Thm. 2.5, let \mathcal{B} be the space of Riemannian metrics on a spin manifold M , and let \mathcal{C} be the space of pairs (∇, h) , where ∇ is a connection on a fixed complex bundle $E \rightarrow M$ and h is a compatible hermitian metric on E . Then there is a “universal index form” U on $LM \times \mathcal{B} \times \mathcal{C}$ such that for each S^1 action a on $(E, \nabla, h) \rightarrow M$ and Riemannian metric g on M for which the action is via isometries, there is an embedding $j = j_{(a, g, \nabla, h)} : M \rightarrow LM \times \mathcal{B} \times \mathcal{C}$ such that the S^1 -index of the twisted Dirac operator is given by $\text{ind}_{S^1} \not{D}_{\nabla^E} = \int_{j_*[M]}^{S^1} U$. In §3, we use the relationship between a^L and a^D and some Kähler geometry to sketch the results on $\pi_1(\text{Diff}(\overline{M}_k))$.

We would like to think that this work touches on several topics that appeared in Prof. Kobayashi’s work: transformation groups (although only S^1 actions for us), and the interplay of Riemannian and complex geometry.

We were privileged to have known Prof. Kobayashi for many years. The second author was a graduate student at Berkeley when Prof. Kobayashi was department chair. At that time,

the math department was in a turf war with another department over office space. Although graduate students in a large department had little direct contact with the chair, letters between Prof. Kobayashi and the administration were regularly posted in the mailroom. In contrast to the typical American style of aggressively defending our territory against intruders, Prof. Kobayashi's letters said in so many words that he would like to give offices to the other department but regretfully could not. The reasons preventing the handover were always very complicated. This tactic seemed to confound the administration, whose puzzled replies took longer and longer to appear in the mailroom and finally ceased altogether. Already from this first encounter, which only involved reading letters, Prof. Kobayashi's gentle determination and sly humor were apparent. Twenty years later, it was a great pleasure to re-encounter Prof. Kobayashi in Japan and to see that his mathematical mind and personality were unchanged.

2. LEADING ORDER CLASSES AND APPLICATIONS

2.1. Infinite rank bundles. Any discussion of infinite rank bundles involves some initial technicalities, just because infinite dimensional vector spaces have many inequivalent norm topologies. In particular, the topologies on smooth functions on a compact manifold associated to different Sobolev norms are inequivalent.

Thus we first have to decide which vector space to use as the model for the fiber of an infinite rank vector bundle $\mathcal{E} \rightarrow \mathcal{M}$ over a paracompact base. Based on the examples in the introduction, we choose fibers modeled on $\Gamma(E)$, where $E \rightarrow M$ is a fixed finite rank complex vector bundle over a closed, oriented manifold. It is important to specify which sections are allowed. From a Hilbert space point of view, it is easiest to work with L^2 sections, but of course such sections have no regularity. In contrast, working with smooth sections forces us to deal with F  chet spaces as fibers; since these spaces are tame in the sense of Hamilton, this is workable but more difficult. As a reasonable compromise, we usually work with the Sobolev space $\mathcal{H} = H^s(E)$ of H^s sections for $s \gg 0$, as these sections are highly differentiable and form a Hilbert space.

We now have to decide on the structure group of \mathcal{E} . The first natural choice of $GL(\mathcal{H})$, the group of bounded automorphisms of \mathcal{H} with bounded inverse, is too large: $GL(\mathcal{H})$ is contractible, so every $GL(\mathcal{H})$ -bundle is trivial. Fortunately, in the cases we consider, the transition functions lie in a gauge group or group of Ψ DOs which have nontrivial topology.

To develop the analog of finite dimensional Chern-Weil theory for, say, the gauge group $\text{Aut}(E)$, we need (i) an Ad-invariant analytic function P on $\text{End}(E) = \text{Lie}(\text{Aut}(E))$, and (ii) an $\text{Aut}(E)$ -connection ∇ on \mathcal{E} . This data will give a characteristic class $[P(\Omega)] \in H^*_{dR}(\mathcal{M}, \mathbb{C})$, where Ω is the curvature of ∇ . The same procedure works for a structure group of Ψ DOs.

The determination of all invariant polynomials or analytic functions on $\text{End}(E)$ is an interesting, perhaps difficult, infinite dimensional version of classical invariant theory. To avoid this issue, we recall that the polynomials $A \mapsto \text{tr}(A^k)$ generate the invariant polynomials on $\mathfrak{u}(n)$. By the same arguments, any trace on $\text{End}(E)$, i.e. a linear map $T : \text{End}(E) \rightarrow \mathbb{C}$ with $T[A, B] = 0$, will give characteristic classes $[T(\Omega^k)]$. The set of all traces is $HH^0(\text{End}(E))$, the zeroth Hochschild cohomology group, which should be computable. In any case, it is somewhat of a relief that the (nonlocal, not computable) operator trace is not a trace on $\text{End}(E)$, since e.g. Id is not trace class. Sidestepping again, we note that

$$A \mapsto \int_M \text{tr}(A) \text{dvol}$$

is a trace on $\text{End}(E)$, where tr is the usual matrix trace and we have fixed a Riemannian metric on M . Varying the metric presumably yields an infinite dimensional vector space of traces, but they are all of the same fundamental type. Moreover, for $f \in C^\infty(M)$, $A \mapsto \int_M \text{tr}(A) f \text{dvol}$ is a trace, and in fact for any distribution $D \in \mathcal{D}'(M)$, $A \mapsto D(\text{tr}(A))$ is a trace. By the time we include distributions, locality is lost, so we will stick to the basic example:

Definition 2.1. *The k^{th} component of the leading order Chern character of the $\text{Aut}(E)$ -bundle $\mathcal{E} \rightarrow \mathcal{M}$ is the de Rham class*

$$c_k^{\text{lo}}(\mathcal{E}) = \frac{1}{k!} \left[\int_M \text{tr}(\Omega^k) \text{dvol} \right] \in H^{2k}(\mathcal{M}),$$

where Ω is the curvature of an $\text{Aut}(E)$ -connection on \mathcal{E} . The leading order Chern character is $ch^{\text{lo}}(\mathcal{E}) = \sum_k c_k^{\text{lo}}(\mathcal{E})$.

We can similarly define leading order Chern classes.

2.2. Leading order classes and mapping spaces. It is well known that the tangent bundle $T\text{Maps}(N, M)$ is the pushdown of a finite rank bundle, as we now explain. At a fixed $f \in \text{Maps}(N, M)$, take a curve $\eta(t) \in \text{Maps}(N, M)$ with $\eta(0) = f$. For each $n \in N$, $\dot{\eta}(t)(n) \in T_{f(n)}M$ gives the infinitesimal information in η at n . Thus an element of $T_f\text{Maps}(N, M)$ is a section $x \mapsto \dot{\eta}(t)(x)$ of $f^*TM \rightarrow N$, so $T_f\text{Maps}(N, M) = \Gamma(f^*TN)$, where we take all smooth sections for the moment. This is summarized in the diagram

$$\begin{array}{ccc}
 \text{ev}^* TM & \longrightarrow & TM \\
 \downarrow & & \downarrow \\
 \text{Maps}(N, M) \times N & \xrightarrow{\text{ev}} & M \\
 \pi \downarrow & & \\
 T\text{Maps}(N, M) = \pi_* \text{ev}^* TM & \longrightarrow & \text{Maps}(N, M)
 \end{array} \tag{2.1}$$

where $\text{ev} : \text{Maps}(N, M) \times N \longrightarrow M$ is the evaluation map $\text{ev}(f, n) = f(n)$, π is the projection, and π_* is the pushdown functor in sheaf theory: $(\pi_* \text{ev}^* TM)|_f = \Gamma(\text{ev}^* TM|_{\pi^{-1}(f)}) = \Gamma(f^* TM)$.

For g in the connected component $T\text{Maps}(N, M)_f$ of f , $g^* TM$ is noncanonically isomorphic to $f^* TM$, and so the sections of these bundles are noncanonically isomorphic. In a neighborhood of f , we can choose these isomorphisms smoothly. This gives a local trivialization of $T\text{Maps}(N, M)$ and implies that on overlaps, the transition functions are given by gauge transformations of the model fiber $\Gamma(f^* TM)$.

Similarly, a bundle $E \longrightarrow M$ induces a bundle $\mathcal{E} = \pi_* \text{ev}^* E \longrightarrow \text{Maps}(N, M)$. The Chern classes of E are related to the leading order classes of \mathcal{E} , as we now show; this has only been sketched before, and the proof below is joint work with A. Larraín-Hubach.

For $n \in N$, let $\text{ev}_n : \text{Maps}(N, M) \longrightarrow N$ be $\text{ev}_n(f) = f(n)$. As n varies over N and f is fixed, $\text{ev}_n^* E$ glues together to a bundle over N which is precisely $f^* E$. Geometrically, for ∇ a connection on E with curvature Ω , $f^* E \longrightarrow N$ has the connection $f^* \nabla$ with curvature $f^* \Omega$ which at n equals $\text{ev}_n^* \Omega$ at $f \in \text{Maps}(N, M)$.

Proposition 2.1. *Assume N is connected and fix $n_0 \in N$. Let $E \longrightarrow M$ be a complex vector bundle and set $\mathcal{E} = \pi_* \text{ev}^* E$. Then*

$$c_k^{\text{lo}}(\mathcal{E}) = \text{vol}(N) \text{ev}_{n_0}^* c_k(E) \in H^{2k}(\text{Maps}(N, M), \mathbb{C}).$$

Proof. We have

$$\begin{aligned}
 c_k^{\text{lo}}(\mathcal{E})(f) &= \int_N \text{tr}(f^*(\Omega^k)) \, d\text{vol}_N \\
 &= \int_N \text{tr}(\text{ev}_n^*(\Omega)^k) \, d\text{vol}_N(n) \\
 &= \int_N \text{ev}_n^* \text{tr}(\Omega^k) \, d\text{vol}_N(n).
 \end{aligned}$$

Let $\gamma_n : [0, 1] \longrightarrow N$ be a path from n to n_0 . For n not in the cut locus \mathcal{C}_{n_0} of n_0 , we can choose γ_n so that $\gamma_n(t)$ is smooth in n and t . Set

$$F : (N \setminus \mathcal{C}_{n_0}) \times \text{Maps}(N, M) \times [0, 1] \longrightarrow M, \quad F(n, f, t) = \text{ev}_{\gamma_n(t)}(f) = f(\gamma_n(t)).$$

$F_n = F|_{(n, \cdot, \cdot)}$ is a homotopy from ev_n to ev_{n_0} depending smoothly on n , so there is a chain homotopy I_n with

$$\text{ev}_n^* \text{tr}(\Omega^k) - \text{ev}_{n_0}^* \text{tr}(\Omega^k) = (d_{\text{Maps}} I_n + I_n d_{\text{Maps} \times [0,1]}) \text{tr} F_n^*(\Omega^k) \quad (2.2)$$

on $\Lambda^{2k}(\text{Maps}(N, M))$. Thus for $N' = N \setminus \mathcal{C}_{n_0}$,

$$\begin{aligned} \int_N \text{ev}_n^* \text{tr}(\Omega^k) \, \text{dvol}_N(n) &= \int_{N'} \text{ev}_n^* \text{tr}(\Omega^k) \, \text{dvol}_N(n) \\ &= \int_{N'} \text{ev}_{n_0}^* \text{tr}[(\Omega^k) + (d_{\text{Maps}} I_n + I_n d_{\text{Maps} \times [0,1]}) \text{tr} F_n^*(\Omega^k)] \, \text{dvol}_N(n), \end{aligned}$$

since the cut locus has measure zero in N . The last integrand, pulled back to the interior of the cut locus in $T_{n_0} N$, extends continuously to the cut locus in $T_{n_0} N$, so the integral is well defined.

Using $d \text{tr}(\alpha) = \text{tr}(\nabla \alpha)$ for a Lie algebra valued form α and setting $\tilde{d} = d_{\text{Maps} \times [0,1]}$ gives

$$\tilde{d} \text{tr} F_n^*(\Omega^k) = \tilde{d} F_n^* \text{tr}(\Omega^k) = F_n^* d_M \text{tr}(\Omega^k) = F_n^* \text{tr}(\nabla(\Omega^k)) = 0,$$

by the Bianchi identity. Thus

$$\begin{aligned} \int_N \text{ev}_n^*(\text{tr}(\Omega^k)) \, \text{dvol}_N(n) &= \int_{N'} \text{ev}_{n_0}^* \text{tr}(\Omega^k) \, \text{dvol}_N(n) + d_{\text{Maps}} \int_{N'} I \text{tr} F_n^*(\Omega^k) \, \text{dvol}_N(n) \\ &= \text{vol}(N) \text{ev}_{n_0}^* \text{tr}(\Omega^k) + d_{\text{Maps}} \int_{N'} I \text{tr} F_n^*(\Omega^k) \, \text{dvol}_N(n). \end{aligned}$$

Therefore

$$c_k^{\text{lo}}(E\mathcal{G}) = \left[\int_N \text{ev}_n^* \text{tr}(\Omega^k) \, \text{dvol}_N(n) \right] = \text{vol}(N) [\text{ev}_{n_0}^* \text{tr}(\Omega^k)] = \text{vol}(N) \text{ev}_{n_0}^* c_k(E).$$

□

By [9, Thm. 4.7], this lemma allows us to detect classes in $H^*(\text{Maps}(N, M), \mathbb{C})$.

Theorem 2.2. *If $E \rightarrow M$ has $c_k(E) \neq 0$, then $c_k^{\text{lo}}(\mathcal{E}) \neq 0$ in $H^*(\text{Maps}(N, M), \mathbb{C})$ for any N .*

The case $N = S^1$ is already important, as the cohomology of LM is given by a cyclic complex construction based on Chen's iterated integral [7]. It would be interesting to relate these two approaches.

2.3. Leading order classes and index theory.

See the note at the beginning of the Introduction.

The S^1 -Atiyah-Singer index theorem can be restated on loop space in a way that handles all isometric actions in one setting. This material extends work in [8].

For a review, assume M is a closed, oriented, Riemannian manifold which is spin and has an S^1 action via isometries. S^1 is also assumed to act on $(E, \nabla^E, h) \rightarrow M$ covering its action on M , where ∇ is an equivariant connection which is hermitian for the hermitian inner product h . In this setup, the kernel and cokernel of the twisted Dirac operator \not{D}_{∇^E} are representations of $S^1 = U(1)$. The S^1 -index of \not{D}_{∇^E} is the corresponding element of the representation ring $R(S^1)$:

$$\text{ind}_{S^1}(\not{D}_{\nabla^E}) = \sum (a_k^+ - a_k^-) t^k \in \mathbb{Z}[t, t^{-1}] = R(S^1),$$

where t^k denotes the representation $e^{i\theta} \mapsto e^{ik\theta}$ of S^1 on \mathbb{C} , and a_{\pm}^k are the multiplicities of t^k in the kernel and cokernel of \not{D}_{∇^E} . Equivalently, for each $e^{i\theta} \in S^1$, we define $\text{ind}_{S^1}(e^{i\theta}, \not{D}_{\nabla^E}) = \sum (a_k^+ - a_k^-) e^{ik\theta}$.

The Atiyah-Segal-Singer fixed point formula computes $\text{ind}_{S^1}(e^{i\theta}, \not{D}_{\nabla^E})$ in terms of data on the fixed point set of a particular $e^{-i\theta}$. (The minus sign is for convenience.) As in [1, Ch. 8], this can be rewritten as

$$\text{ind}_{S^1}(e^{-i\theta}, \not{D}_{\nabla^E}) = (2\pi i)^{-\dim(M)/2} \int_M \hat{A}_u(\theta, \Omega_u) \text{ch}(\theta, \Omega_u^E). \quad (2.3)$$

Here Ω_u is the equivariant curvature of the Levi-Civita connection on M , and $\hat{A}_u(\theta, \Omega_u) \in \Lambda^*(M)$ is the equivariant \hat{A} -polynomial $\hat{A}(\Omega_u) \in (\mathbb{C}[u] \otimes \Lambda^* M)^{S^1}$ evaluated at $\theta \in \mathfrak{u}(1)$. Similarly, Ω_u^E is the equivariant curvature of ∇^E , and ch denotes the equivariant Chern character. We condense the notation in (2.3) to

$$\overline{\text{ind}}_{S^1}(\not{D}_{\nabla^E}) = (2\pi i)^{-\dim(M)/2} \int_M^{S^1} \hat{A}_u(\Omega_u) \text{ch}(\Omega_u^E), \quad (2.4)$$

with the bar on the left hand side indicating evaluation at $e^{-i\theta}$ and the right hand side evaluated at θ . For the trivial action, (2.4) is exactly the ordinary index theorem.

As mentioned in the Introduction, the circle action $a : S^1 \times M \rightarrow M$ induces $a^L : M \rightarrow LM$, $a^L(m)(\theta) = a(\theta, m)$. Denoting a^L just by a , we get the class $a_*[M] \in H_n(LM, \mathbb{Z})$ determined by the action. LM has the rotation action $r : S^1 \times LM \rightarrow LM$, $r(\theta, \gamma)(\psi) = \gamma(\theta + \psi)$. This action is via isometries for the L^2 metric on LM :

$$\langle X, Y \rangle_\gamma = \frac{1}{2\pi} \int_{S^1} \langle X(\theta), Y(\theta) \rangle_{\gamma(\theta)} d\theta, \quad (2.5)$$

for $X, Y \in T_\gamma LM = \Gamma(\gamma^* TM \rightarrow S^1)$.¹ The suitably averaged L^2 Levi-Civita connection is r -equivariant, so we can form the equivariant curvature $\tilde{\Omega}_u$ and $\hat{A}(\tilde{\Omega}_u)$ on LM . The \hat{A} -polynomial is an equivariant form on LM , denoted $\hat{A}(\tilde{\Omega}_u) \in (\mathbb{C}[u] \otimes \Lambda^*(LM))^{S^1}$. Similarly, we can take the

¹Recall that we take H^s sections, so this is a weak Riemannian metric on LM .

equivariant curvature Ω_u^E of ∇ , form the L^2 weak connection on $\mathcal{E} = \pi_* \text{ev}^* E \rightarrow LM$, average it to form the equivariant curvature $\tilde{\Omega}_u^\mathcal{E}$, and then take its equivariant Chern character $\text{ch}(\tilde{\Omega}_u^\mathcal{E})$.

The map $a = a^L : M \rightarrow LM$ easily intertwines the actions a on $E \rightarrow M$ and r on $\mathcal{E} \rightarrow LM$. From this, we easily get

$$a^* \hat{A}(\tilde{\Omega}_u) = \hat{A}(\Omega_u), \quad a^* \text{ch}(\tilde{\Omega}_u^\mathcal{E}) = \text{ch}(\Omega_u^E).$$

From the S^1 -index theorem, we therefore obtain:

Theorem 2.3. *Let M be a compact, oriented Riemannian spin manifold with an isometric S^1 -action, and let E be an equivariant hermitian bundle with connection ∇^E over M . Then*

$$\overline{\text{ind}}_{S^1}(\not{D}_{\nabla^E}) = (2\pi i)^{-\dim(M)/2} \int_{a_*[M]}^{S^1} \hat{A}(\tilde{\Omega}_u) \text{ch}(\tilde{\Omega}_u^\mathcal{E}).$$

It is easy to check that on the copy of M sitting inside LM as the constant loops, $\hat{A}(\tilde{\Omega}_u), \text{ch}(\tilde{\Omega}_u^\mathcal{E})$ reduce to the \hat{A} -polynomial of M and the Chern character of E , respectively. Thus these forms give equivariantly closed extensions of these characteristic classes to LM . A very different construction of an equivariant Chern character on LM is given in [22], based on ideas in [2, 7]. It is natural to conjecture that the equivariant classes of the two Chern characters are the same; see [15] for preliminary results.

The important point of Thm. 2.3 is that the action information on the right hand side is now contained in the “action class” $a_*[M]$, while the integrand only depends on the metrics on M, E . In particular, the integrand is applicable to all isometric actions for fixed metrics.

In fact, we can remove this metric dependence from the integrand as follows. Let \mathcal{B} be the space of metrics on M . \mathcal{B} comes with a natural Riemannian L^2 metric $g^\mathcal{B}$, given at $T_{g_0}\mathcal{B}$ by

$$g^\mathcal{B}(X, Y) = \int_M g_0^{ab} g_0^{cd} X_{ac} Y_{bd} \text{dvol}_{g_0}.$$

Here $X = X_{ac} dx^a \otimes dx^c \in T_{g_0}\mathcal{B}$ and similarly for Y . Thus $LM \times \mathcal{B}$ has a metric h which at (γ, g_0) is the non-product metric determined by the L^2 metric on $T_\gamma LM$ given by g_0 and by $g^\mathcal{B}$ on $T_{g_0}\mathcal{B}$. We extend the rotational action on LM trivially to $LM \times \mathcal{B}$, and so obtain an equivariant curvature $\tilde{F}_u \in \Lambda^*(LM \times \mathcal{B}, \text{End}(TLM \oplus T\mathcal{B}))$. Let $i_{g_0} : LM \rightarrow LM \times \mathcal{B}$ be the inclusion $\gamma \mapsto (\gamma, g_0)$. By $i_{g_0}^* \tilde{F}_u$, we mean that we restrict \tilde{F}_u to tangent vectors in TLM , and the “endomorphism part” A of \tilde{F}_u is replaced by $P^{TLM} A P^{TLM}$, where P^{TLM} is the h -orthogonal projection of $T(LM \times \mathcal{B})$ to TLM . (Here we put the Sobolev topology on LM and \mathcal{B} for some high Sobolev parameter, so that TLM is closed in $T(LM \times \mathcal{B})$.)

Theorem 2.4. (i) $i_{g_0}^* \tilde{F}_u = \tilde{\Omega}_u^{g_0}$, where $\tilde{\Omega}_u^{g_0}$ is $\tilde{\Omega}_u$ computed at the metric g_0 .

(ii) If a is a g_0 -invariant S^1 action on M , then

$$\overline{\text{ind}}_{S^1}(\not\partial) = (2\pi i)^{-\dim(M)/2} \int_{i_{g_0,*}a_*[M]}^{S^1} \hat{A}(\tilde{F}_u).$$

Since every S^1 action is isometric for some metric, this produces a *universal equivariant \hat{A} -polynomial* $\hat{A}(\tilde{\Omega}_u)$ on $LM \times \mathcal{B}$, i.e., an equivariantly closed form on $LM \times \mathcal{B}$ such that the S^1 -index is determined by (i) the universal \hat{A} -polynomial and (ii) the cycle of integration associated to the action and compatible metric.

Proof. (i) The Levi-Civita connection on a Riemannian Hilbert manifold is characterized by

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle &= X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle \\ &\quad + \langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle - \langle [Y, Z], X \rangle. \end{aligned} \quad (2.6)$$

We use the right hand side of (2.6) with the L^2 inner product to motivate the definition of a connection $\tilde{\nabla}$ on $LM \times \mathcal{B}$, avoiding the issue of determining the topology for which the right hand side is a continuous linear functional of Z .

On $LM \times \mathcal{B}$, we have at (γ, g)

$$(X, T)\langle (Y, S), (Z, R) \rangle = X\langle Y, Z \rangle + T\langle S, R \rangle + \frac{1}{2\pi} \int_{S^1} \langle Y(\theta), Z(\theta) \rangle_T d\theta$$

using $\delta_T g = T$. Here $\langle Y(\theta), Z(\theta) \rangle_T = T_{ab}(\gamma(\theta))Y^a(\theta)Z^b(\theta)$. We also have $\langle [(X, T), (Y, S)], (Z, R) \rangle = \langle [X, Y], Z \rangle + \langle [T, S], R \rangle$. Therefore we must have

$$\langle \tilde{\nabla}_{(X,T)}(Y, S), (Z, R) \rangle = \langle \nabla_X^{LM} Y, Z \rangle + \langle \nabla_T^{\mathcal{B}} S, R \rangle + \frac{1}{2} \alpha^\sharp, \quad (2.7)$$

where α^\sharp is the tangent vector L^2 dual to the one-form on $LM \times \mathcal{B}$ given by

$$(Z, R) \mapsto \langle Y, Z \rangle_T + \langle X, Z \rangle_S - \langle X, Y \rangle_R,$$

and we use the shorthand $\langle Y, Z \rangle_T = \frac{1}{2\pi} \int_{S^1} \langle Y(\theta), Z(\theta) \rangle_T d\theta$. Since $\int_\gamma \omega = \int_M \omega \wedge \text{PD}(\gamma)$ for one-forms ω (where for the Poincaré dual $\text{PD}(\gamma)$ we may assume that γ is embedded²), we have

$$\begin{aligned} \langle X, Y \rangle_R &= \frac{1}{2\pi} \int_{S^1} R_{ab} X^a Y^b d\theta &= \frac{1}{2\pi} \int_M R_{ab} X^a Y^b d\theta \wedge \text{PD}(\gamma) \\ &= \int_M g^{ac} g^{bd} R_{ab} X_c Y_d * (d\theta \wedge \text{PD}(\gamma)) \text{dvol} \\ &= \langle R, *(d\theta \wedge \text{PD}(\gamma)) X^\flat \hat{\otimes} Y^\flat \rangle, \end{aligned}$$

²By a small perturbation of γ , at least if $\dim(M) > 2$.

where $X^b \hat{\otimes} Y^b$ is the symmetric product of the one-forms $X^b = X_c dx^c, Y^b = Y_d dx^d$ dual to X, Y . We also have

$$\begin{aligned} \langle Y, Z \rangle_T &= \frac{1}{2\pi} \int_{S^1} T_{ab} Y^a Z^b = \frac{1}{2\pi} \int_{S^1} g_{bd} g_{ac} T^{cd} Y^a Z^b \\ &= \langle Z, g_{ac} Y^a T^{cd} \partial_d \rangle = \langle Z, Y^a T_a{}^d \partial_d \rangle \\ &= \langle Z, i_Y T \rangle, \end{aligned}$$

with i_Y the interior product. Similarly, $\langle X, Z \rangle_S = \langle Z, i_X S \rangle$. We obtain that the connection $\tilde{\nabla}$ on $LM \times \mathcal{B}$ must be

$$\begin{aligned} \tilde{\nabla}_{(X,T)}(Y, S)_{(\gamma, g)} & \\ &= (\nabla_X^{LM} Y, 0) + (0, \nabla_T^{\mathcal{B}} S) + \frac{1}{2} [((i_Y T)^\sharp, 0) + ((i_X S)^\sharp, 0) - (0, *(d\theta \wedge \text{PD}(\gamma)) X^b \hat{\otimes} Y^b)], \end{aligned} \quad (2.8)$$

where ∇^{LM} is computed for g . Since the last term on the right hand side of (2.8) is an endomorphism of (Y, S) , $\tilde{\nabla}$ is a connection. ($\tilde{\nabla}$ is derived from (2.6), so it is a torsion free, metric connection for the L^2 metric. However, (2.8) does not make sense for $(X, T), (Y, S)$ in L^2 , so we cannot claim that $\tilde{\nabla}$ is the L^2 Levi-Civita connection.)

Denoting $(X, 0) \in T(LM \times \mathcal{B})$ by X , the curvature \tilde{F} of $\tilde{\nabla}$ restricted to $LM \times \{g_0\}$ is $i_{g_0}^* \tilde{F} = P^{TLM}(\tilde{\nabla}_X \tilde{\nabla}_Y - \tilde{\nabla}_Y \tilde{\nabla}_X - \tilde{\nabla}_{[X, Y]}) P^{TLM}$. We have

$$\begin{aligned} P^{TLM} \tilde{\nabla}_X \tilde{\nabla}_Y Z &= P^{TLM} \tilde{\nabla}_X [(\nabla_Y^{LM} Z, 0) - \frac{1}{2} (0, *(d\theta \wedge \text{PD}(\gamma)) Y^b \hat{\otimes} Z^b)] \\ &= P^{TLM} [(\nabla_X^{LM} \nabla_Y^{LM} Z, 0) - \frac{1}{2} (0, *(d\theta \wedge \text{PD}(\gamma)) X^b \hat{\otimes} (\nabla_Y^{LM} Z)^b) + \frac{1}{2} ([i_X(Y^b \hat{\otimes} Z^b)]^\sharp, 0)] \\ &= \nabla_X^{LM} \nabla_Y^{LM} Z + \frac{1}{2} \langle X, Y \rangle Z. \end{aligned}$$

Since $P^{TLM} \tilde{\nabla}_{[X, Y]} Z = \nabla_{[X, Y]}^{LM} Z$, we get $i_{g_0}^* \tilde{F} = \tilde{\Omega}$.

The equivariant curvature on $LM \times \mathcal{B}$ is given by $\tilde{F}_u = \tilde{F} + \mu$, with $\mu(\dot{\gamma}, 0) = \mathcal{L}_{(\dot{\gamma}, 0)} - \tilde{\nabla}_{(\dot{\gamma}, 0)}$, since $(\dot{\gamma}, 0)$ is the vector field of the action. It follows that $P^{TLM} \mu_{(\dot{\gamma}, 0)} = \mathcal{L}_{\dot{\gamma}} - \nabla_{\dot{\gamma}}^{LM} = \mu_{\dot{\gamma}}$. Thus $i_{g_0}^* \tilde{F}_u = \tilde{\Omega}_u$.

(ii) By Thm 2.3, we have

$$(2\pi i)^{\dim(M)/2} \overline{\text{ind}}_{S^1}(\partial) = \int_{a_*[M]}^{S^1} \hat{A}(\tilde{\Omega}_u) = \int_{a_*[M]}^{S^1} \hat{A}(i_{g_0}^* \tilde{F}_u) = \int_{a_*[M]}^{S^1} i_{g_0}^* \hat{A}(\tilde{F}_u) = \int_{i_{g_0, *a_*[M]}^{S^1}} \hat{A}(\tilde{F}_u).$$

□

We now sketch the easier construction of a universal Chern character for a fixed bundle $E \rightarrow M$. Let $\mathcal{C} = \{(\nabla^E, h^E)\}$, where ∇^E is a hermitian connection on E for the hermitian metric h^E . \mathcal{C} fibers over \mathcal{H} , the space of hermitian inner products on E , with fiber \mathcal{C}_h modeled on $\Lambda^1(M, \text{End}_h(E))$,

with $\text{End}_h(E)$ the space of h -skew-hermitian endomorphisms of E . This fibration is locally trivial: (i) There is a coset H of the h -unitary frame bundle of E inside the bundle of $GL(n, \mathbb{C})$ -frames such that $A \in H$ iff A takes an h -orthonormal frame of E to an h' -orthonormal frame; (ii) C is h' -skew-hermitian iff ACA^{-1} is h -skew-hermitian; (iii) For all h' close to h , we can make a smooth choice of $A = A_{h'}$ in a contractible neighborhood of the identity in $\Gamma(\text{End}(E))$, giving smoothly varying isomorphisms $\Lambda^1(M, \text{End}_h(E)) \simeq \Lambda^1(M, \text{End}_{h'}(E))$. Thus \mathcal{C} is a Banach or Fréchet manifold, a subset of $\mathcal{H} \times \mathcal{A}$, where \mathcal{A} is the space of connections on E .

The bundle $\mathcal{E} = \pi_* \text{ev}^* E \rightarrow LM$ pulls back to $p^* \mathcal{E} \rightarrow LM \times \mathcal{C}$ under the projection $p : LM \times \mathcal{C} \rightarrow LM$. $p^* E$ has the connection given at (γ, ∇, h) by

$$\widehat{\nabla}_{(X, \omega, T)} s = \widetilde{\nabla}_X^\nabla s + \delta_\omega s + \delta_T s,$$

where $\widetilde{\nabla}^\nabla$ is the connection on \mathcal{E} associated to ∇ , and δ_ω, δ_T denote trivial connections in the \mathcal{A}, \mathcal{H} directions. For $\hat{X}_i = (X_i, 0, 0)$, it follows immediately that $\widehat{\Omega}(\hat{X}_1, \hat{X}_2)_{(\gamma, \nabla, h)} = \widetilde{\Omega}^\nabla(X_1, X_2)_\gamma$ in the obvious notation. For $\widetilde{\Omega}^\nabla = (\widetilde{\Omega}^\nabla)_{ij}{}^a{}_b dx^i \wedge dx^j \otimes e^a \otimes e_b$, for $\{e_b\}$ a local frame of E with dual frame $\{e^a\}$, we define

$$\begin{aligned} \text{Tr}(\widehat{\Omega})_{(\gamma, \nabla, h)} &= \left(\frac{1}{2\pi} \int_{S^1} h_{ac} h^{cb} [(\widehat{\Omega}^\nabla)_{ij}{}^a{}_b]_{(\gamma, \nabla, h)}(\theta) d\theta \right) dx^i \wedge dx^j \\ &= \left(\frac{1}{2\pi} \int_{S^1} h_{ac} h^{cb} (\widetilde{\Omega}^\nabla)_{ij}{}^a{}_b(\gamma)(\theta) d\theta \right) dx^i \wedge dx^j \\ &= \left(\frac{1}{2\pi} \int_{S^1} h_{ac} h^{cb} (\Omega^\nabla)_{ij}{}^a{}_b(\gamma(\theta)) d\theta \right) dx^i \wedge dx^j. \end{aligned}$$

The same result holds for $\widehat{\Omega}_u$. $\text{Tr}(\widehat{\Omega}_u^k)$ and $\text{ch}(\widehat{\Omega}_u)$ are defined similarly. Let (∇, h) be an equivariant connection and hermitian metric for an action of S^1 on E . For the inclusion $j : LM \rightarrow LM \times \mathcal{C}$, $\gamma \mapsto (\gamma, \nabla, h)$, we have $j^* \text{ch}(\widehat{\Omega}_u) = \text{ch}(\widetilde{\Omega}_u^\nabla)$, where $\widetilde{\Omega}_u^\nabla$ is the equivariant curvature of \mathcal{E} associated to ∇ . Thus $\text{ch}(\widehat{\Omega}_u)$ is a universal equivariant Chern character for $E \rightarrow M$.

Finally, one can combine the equivariantly closed forms $\text{ch}(\widehat{\Omega}_u) \in (\mathbb{C}[u] \otimes \Lambda^*(M \times \mathcal{C}))^{S^1}$ with the universal \hat{A} -form in $(\mathbb{C}[u] \otimes \Lambda^*(LM \times \mathcal{B}))^{S^1}$ by pulling them back to $(\mathbb{C}[u] \otimes \Lambda^*(LM \times \mathcal{B} \times \mathcal{C}))^{S^1}$. In summary:

Theorem 2.5. *(i) Let (M, g) have an isometric S^1 action and let $(E, \nabla, h) \rightarrow M$ be an equivariant bundle with an h -hermitian equivariant connection ∇ . Let $j = j_{(g, \nabla, h)} : LM \rightarrow LM \times \mathcal{B} \times \mathcal{C}$ be the injection $j(\gamma) = (\gamma, g, \nabla, h)$. The equivariantly closed form*

$$\hat{A}(\widetilde{F}_u) \text{ch}(\widehat{\Omega}_u) \in (\mathbb{C}[u] \otimes \Lambda^*(LM \times \mathcal{B} \times \mathcal{C}))^{S^1}$$

has $j^[\hat{A}(\widetilde{F}_u) \text{ch}(\widehat{\Omega}_u)] = \hat{A}(\widetilde{\Omega}_u^g) \text{ch}(\widetilde{\Omega}_u^\nabla)$.*

(ii) We have

$$\overline{\text{ind}}_{S^1}(\not\partial_{\nabla^E}) = (2\pi i)^{-\dim(M)/2} \int_{j_* a_*[M]}^{S^1} \hat{A}(\tilde{F}_u) \text{ch}(\hat{\Omega}_u).$$

Thus the “universal index form” $\hat{A}(\tilde{F}_u) \text{ch}(\hat{\Omega}_u)$ determines the S^1 -index for every action of the circle on $E \rightarrow M$.

2.4. Flat fibrations and Gromov-Witten invariants. The fibration π in (2.1) is trivial. In this subsection, we discuss to what extent leading order classes appear in nontrivial fibrations and give applications to Gromov-Witten theory.

A finite rank bundle $E \rightarrow M$ on the total space of a fibration $Z \rightarrow M \xrightarrow{\pi} B$ of manifolds gives rise to an infinite rank bundle $\mathcal{E} = \pi_* E \rightarrow B$ with $\mathcal{E}_b = \Gamma(E|_{\pi^{-1}(b)})$. Fix a connection D for the fibration, *i.e.*, a complement to the kernel of π_* in TM . The connection ∇ pushes down to a connection $\pi_* \nabla = \nabla'$ on \mathcal{E} by

$$\pi_* \nabla_X(s')(m) = \nabla_{X^h}(\tilde{s})(m), \quad (2.9)$$

where X^h is the D -horizontal lift of $X \in T_b B$ to $T_m M$, $s' \in \Gamma(\mathcal{E})$, and $\tilde{s} \in \Gamma(E)$ is defined by $\tilde{s}(m) = s'(\pi(m))(m)$. The curvature Ω' of ∇' satisfies

$$\begin{aligned} \Omega'(X, Y) &= \pi_* \nabla_X \pi_* \nabla_Y - \pi_* \nabla_Y \pi_* \nabla_X - \pi_* \nabla_{[X, Y]} \\ &= \nabla_{X^h} \nabla_{Y^h} - \nabla_{Y^h} \nabla_{X^h} - \nabla_{[X, Y]^h} \\ &= \nabla_{X^h} \nabla_{Y^h} - \nabla_{Y^h} \nabla_{X^h} - \nabla_{[X^h, Y^h]} + (\nabla_{[X^h, Y^h]} - \nabla_{[X, Y]^h}) \\ &= \Omega(X^h, Y^h) + (\nabla_{[X^h, Y^h]} - \nabla_{[X, Y]^h}). \end{aligned}$$

$\Omega(X^h, Y^h)$ is a zeroth order or multiplication operator, so in general, $\nabla_{[X^h, Y^h]} - \nabla_{[X, Y]^h}$ and hence Ω' acts on the fibers of \mathcal{E}_b as a first order differential operator.

The leading order trace is only a trace on differential operators (or Ψ DOs) of nonpositive order. Thus we are naturally led to restrict attention to fibrations with flat or integrable connections, which by definition means $[X^h, Y^h] - [X, Y]^h = 0$. Flat fibrations appear in Gromov-Witten theory and for mapping spaces, but the setup for the families index theorem involves non-flat fibrations; it is a major drawback that our approach does not apply to this case.

We summarize the setup for Gromov-Witten theory, with more details in [8]. Let M be a closed symplectic manifold with a generic compatible almost complex structure. For $A \in H_2(M, \mathbb{Z})$, set $C_0^\infty(A) = \{f : \mathbb{P}^1 \rightarrow M \mid f \in C^\infty, f \text{ simple}, f_*[\mathbb{P}^1] = A\}$. Set $\mathbb{P}_k^1 = \{(x_1, \dots, x_k) \in (\mathbb{P}^1)^k : x_i \neq x_j \text{ for } i \neq j\}$. For fixed $k \in \mathbb{Z}_{\geq 0}$, set

$$C_{0,k}^\infty(A) = (C_0^\infty(A) \times \mathbb{P}_k^1) / \text{Aut}(\mathbb{P}^1).$$

$C_{0,k}^\infty(A)$ is an infinite dimensional manifold of either Banach or Fréchet type. Denoting an element of $C_{0,k}^\infty(A)$ by $[f, x_1, \dots, x_k]$, we set the moduli space of pseudoholomorphic maps to be $\mathcal{M}_{0,k}(A) = \{[f, x_1, \dots, x_k] : f \text{ is pseudoholomorphic}\}$. $\mathcal{M}_{0,k}(A)$ is a smooth, finite dimensional, noncompact manifold.

The forgetful map $\pi = \pi_k : C_{0,k}^\infty(A) \longrightarrow C_{0,k-1}^\infty(A)$ given by $[f, x_1, \dots, x_{k-1}, x_k] \mapsto [f, x_1, \dots, x_{k-1}]$ is a locally trivial smooth fibration. It is shown in [8] that π is flat. As a result, we can relate Gromov-Witten invariants on $\mathcal{M}_{0,k}(A)$ to leading order classes on $\mathcal{M}_{0,k-1}(A)$, at least in the case where the boundary of these moduli spaces is homologically small, i.e., the boundaries of the compactified moduli spaces have big enough codimension. This occurs for M semipositive, e.g. for many smooth projective Fano varieties. In this case, the Gromov-Witten invariants $\langle \alpha_1^{\ell_1} \dots \alpha_k^{\ell_k} \rangle$, for $\alpha_i \in H^*(M, \mathbb{C})$, are given by the expected integral $\int_{\mathcal{M}_{0,k}(A)} \text{ev}^*(\alpha_1^{\ell_1} \wedge \dots \wedge \alpha_k^{\ell_k})$, where $\text{ev}[f, x_1, \dots, x_k] = (f(x_1), \dots, f(x_k))$. This is a very special case, as usually GW invariants involve the virtual fundamental class of the compactified moduli space.

To state a result, let α_i be elements of the even cohomology of M . Since the Chern character $ch : K(M) \otimes \mathbb{C} \longrightarrow H^{\text{ev}}(M, \mathbb{C})$ is an isomorphism, $\alpha_i = ch(E_i)$ for a virtual bundle E_i . Pullbacks and pushdowns of the E_i are well defined virtual bundles. Let $\pi_* ch(E_k)$ be the usual pushdown/integration over the fiber of $ch(E_k)$. In [16], this class is called the string Chern class $ch^{\text{str}}(\mathcal{E}_k)$ of $\mathcal{E}_k = \pi_* \text{ev}^* E_k$. Recall that the leading order Chern character is given in Def. 2.1. Finally, set $E_i^\ell = E_i^{\otimes \ell}$.

Theorem 2.6. *Let $\alpha_i \in H^{\text{ev}}(M, \mathbb{C})$ satisfy $\alpha_i = ch(E_i)$ for $E_i \in K(M)$. Set $\mathcal{E}_i = \pi_* \text{ev}_i^* E_i \longrightarrow \mathcal{M}_{0,k-1}(A)$. Then*

$$\langle \alpha_1^{\ell_1} \dots \alpha_k^{\ell_k} \rangle_{0,k} = \langle ch^{\text{lo}}(\mathcal{E}_1^{\ell_1}) \dots ch^{\text{lo}}(\mathcal{E}_{k-1}^{\ell_{k-1}}) ch^{\text{str}}(\mathcal{E}_k) \rangle_{0,k-1}.$$

GW invariants have been used very successfully to distinguish symplectic structures on manifolds. The leading order classes exist on the larger space $C_{0,k}^\infty(A)$. There may be other symplectically defined cycles in this space that could be used similarly. For example, the moduli spaces are minima for the holomorphic energy functional on $C_{0,k}^\infty(A)$; perhaps moduli spaces of nonminimal critical maps contain new homological information detected by leading order classes.

2.5. Applications to loop groups and Donaldson invariants. We briefly sketch other applications of leading order classes from [8].

Loop groups ΩG are of course a very special mapping space. The generators of $H^*(\Omega G, \mathbb{R})$ for G compact are known [19, §4.11]. As stated below, these generators are equal to certain leading order Chern-Simons classes or equivalently Chern-Simons string classes, which are defined for a

pair of connections just as in finite dimensions. We start with a degree k Ad_G -invariant polynomial on the Lie algebra \mathfrak{g} of G . For $G = U(n)$, f is in the algebra generated by the polarization of $A \mapsto \text{Tr}(A^k)$. Just as with leading order Chern classes, we can associate a leading order class f^{lo} to any pushdown bundle $\mathcal{E} \rightarrow B$, where $E \rightarrow M$ is a G -bundle and $M \rightarrow B$ is a Riemannian fibration. While all this works for principal bundles, to fit with the previous setting of vector bundles, we choose a faithful Lie algebra representation on a finite dimensional vector space V , let $h : G \rightarrow \text{Aut}(V)$ be the exponentiated representation, and work on the associated vector bundle $E \times_h V \rightarrow M$. In particular, in Def. 2.1, we just replace $\text{tr}(\Omega^k)$ with $f(\Omega, \dots, \Omega)$.

Given a pair of connections ∇_0, ∇_1 on $E \rightarrow M$ with connection one-forms ω_0, ω_1 and a Riemannian fibration $Z \rightarrow M \rightarrow B$ with fiber Z_b over b , we define

$$CS_f^{\text{lo}}(\pi_* \nabla_0, \pi_* \nabla_1) = \int_0^1 \int_{Z_b} f((\omega_1 - \omega_0), \Omega_t, \dots, \Omega_t) d\text{vol}_{Z_b} \in \Lambda^{2k-1}(B),$$

with $k-1$ occurrences of Ω_t , where $\Omega_t = d\omega_t + \omega_t \wedge \omega_t$, $\omega_t = t\omega_0 + (1-t)\omega_1$. As usual,

$d_B CS_f^{\text{lo}}(\pi_* \nabla_0, \pi_* \nabla_1) = f^{\text{lo}}(\pi_* \Omega_0) - f^{\text{lo}}(\pi_* \Omega_1)$, so the leading order Chern-Simons forms are closed provided the leading order Chern forms for ∇_0, ∇_1 vanish pointwise.

To build leading order CS classes on ΩG , we use the fibration (2.1) with $N = S^1, M = G$. Let G have Lie algebra \mathfrak{g} and Maurer-Cartan form θ^G . Choose $h : G \rightarrow \text{Aut}(V)$ as above. For $\underline{V} \rightarrow G$ the trivial vector bundle $G \times V \rightarrow G$, we can view h as a gauge transformation of \underline{V} . Let $\nabla_0 = d$ be the trivial connection on \underline{V} , and let $\nabla_1 = h \cdot \nabla_0 = h^{-1}dh$ be the gauge transformed connection. Since the connections are flat, the CS classes $CS_f^{\text{lo}}(\pi_* \nabla_0, \pi_* \nabla_1) \in H^{2k-1}(\Omega G, \mathbb{C})$ are defined. Similarly, CS string classes are given by integrating over the fiber S^1 , so $CS_f^{\text{str}, \mathcal{E}}(\text{ev}^* \nabla_0, \text{ev}^* \nabla_1) = \pi_* CS_f^{\text{ev}^* E}(\text{ev}^* \nabla_0, \text{ev}^* \nabla_1) \in H^{2k-2}(\Omega G, \mathbb{C})$ is defined for $E \rightarrow G$.

To state the results, let χ be the vector field on ΩG associated to the rotation action on loops: $\chi(\gamma)(\theta) = \dot{\gamma}(\theta)$. Let i_χ denote the interior product.

Theorem 2.7. *Let $\mathcal{V} = \pi_* \text{ev}^* \underline{V}$. Then $H^*(\Omega G, \mathbb{R})$ is generated by*

$$CS_{f_i}^{\text{str}, \mathcal{V}}(\text{ev}^* \nabla_0, \text{ev}^* \nabla_1) = i_\chi CS_f^{\text{lo}, \mathcal{V}}(\pi_* \text{ev}^* \nabla_0, \pi_* \text{ev}^* \nabla_1).$$

To describe the relationship between leading order classes and Donaldson invariants, we review the basic setup. Let $P \rightarrow M$ be a principal G -bundle over a closed manifold M for a compact semisimple group G . We denote by \mathcal{A}^* , resp. \mathcal{G} , the space of irreducible connections on P , resp. the gauge group of P . For a connection A on P , let $d_A : \text{Lie}(\mathcal{G}) = \Lambda^0(M, \text{Ad } P) \rightarrow \Lambda^1(M, \text{Ad } P)$ be the covariant derivative associated to A on the adjoint bundle $\text{Ad } P = P \times_{\text{Ad}} \mathfrak{g}$. Then the vertical space of $\mathcal{A}^* \rightarrow \mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ at A is $\text{Im}(d_A)$. The orthogonal complement $\ker d_A^*$ forms the

horizontal space of a connection ω on $\mathcal{A}^* \rightarrow \mathcal{B}^*$. Let Ω be the curvature of this connection. Let $G_A = (d_A^* d_A)^{-1}$ be the Green's operator associated to d_A .

Lemma 2.8. *For X, Y horizontal tangent vectors at A , we have*

$$\Omega_A(X, Y) = -2G_A * [X, *Y] \in \text{Lie}(\mathcal{G}) = \Lambda^0(M, \text{Ad } P).$$

$\text{Lie}(\mathcal{G})$ can be thought of as an algebra of multiplication operators via the injective adjoint representation of \mathfrak{g} . Equivalently, we can pass to the \mathcal{G} -vector bundle $\text{Ad } \mathcal{A}^* = \mathcal{A}^* \times_{\text{Ad}} \text{Lie}(\mathcal{G})$ with fiber $\text{Lie}(\mathcal{G})$ and take the leading order classes $c_k^{\text{lo}, (\text{Ad } \mathcal{A}^*) \otimes \mathbb{C}}$ of its associated connection $d\text{Ad}(\omega)$, whose curvature $[\Omega, \cdot]$ is usually denoted just by Ω . Either way, the leading order Chern form $c_k^{\text{lo}}(\Omega)$ of $\mathcal{A}^* \rightarrow \mathcal{B}^*$ is given by $\int_M \text{tr}(\Omega^{\wedge k}) \text{dvol}$ for some Riemannian metric on M . Here $C^{\wedge k}$ is the endomorphism on $\Lambda^k V$ determined by an endomorphism C on V . Below, we denote $\Omega^{\wedge k}$ by Ω^k , with the caution that this is not the same as the Ω^k occurring in the Chern character.

On 4-manifolds, Donaldson invariants are built from his ν and μ classes in $H^*(\mathcal{M}, \mathbb{Z})$, where \mathcal{M} is the moduli space of self-dual connections. In fact, these classes are constructed on \mathcal{B}^* and then restricted to \mathcal{M} . By comparing with explicit calculations in [4], we get

Proposition 2.9. *As differential forms, ν equals $p_1^{\text{lo}, \mathcal{A}^*}(\Omega) = -c_2^{\text{lo}, (\text{Ad } \mathcal{A}^*) \otimes \mathbb{C}}(\Omega)$ up to a constant.*

For the μ classes, we take $a \in H_2(M, \mathbb{Q})$, and Donaldson's map $\mu : H_*(M, \mathbb{Q}) \rightarrow H^{4-*}(\mathcal{M}, \mathbb{Q})$. Recall that $\mu(a) = i^*(\nu/a)$, for the slant product $\nu/ : H_*(M, \mathbb{Q}) \rightarrow H^{4-*}(\mathcal{B}^*, \mathbb{Q})$ and $i : \mathcal{M} \rightarrow \mathcal{B}^*$ the inclusion. In particular, $\nu = \mu(1)$ for $1 \in H_0(M)$. By [4, Prop. 5.2.18], the two-form $C_\omega \in \Lambda^2(\mathcal{M})$ representing $\nu/a = \nu/\text{PD}^{-1}(\omega)$ and hence $\mu(a)$ is given at $[A] \in \mathcal{M}$ by

$$C_\omega(X, Y) = \frac{1}{8\pi^2} \int_M \text{tr}(X \wedge Y) \wedge \omega + \frac{1}{2\pi^2} \int_M \text{tr}(\Omega_A(X, Y) F_A) \wedge \omega, \quad (2.10)$$

where F_A is the curvature of A . On the right hand side, we use any $A \in [A]$ and $X, Y \in T_A \mathcal{A}^*$ with $d_A^* X = d_A^* Y = 0$.

As mentioned before Def 2.1, there is a leading order class associated to any distribution or zero current Λ on $C^\infty(M)$, given pointwise by $c_k^{\text{lo}, \Lambda} = \Lambda(\text{tr}(\Omega^k))$, where Ω is the curvature of a connection taking values in the Lie algebra of a gauge group, as in this section. In particular, for a fixed $f \in C^\infty(M)$ we have the characteristic class $\int_M f \cdot \text{tr}(\Omega^k)$. We can just as well consider $\text{tr}(\Omega^k)$ as a zero-current acting on f . Looking back at (2.10), we can consider the two-currents

$$\text{tr}(X \wedge Y), \quad \text{tr}(\Omega_A(X, Y) F_A), \quad (2.11)$$

for fixed X, Y . Thus we can consider C as an element of $\Lambda^2(\mathcal{M}, \mathcal{D}^2)$, the space of two-current valued two-forms on \mathcal{M} .

Because these two-currents are Ad_G -invariant, the usual Chern-Weil proof shows that $C(\omega) = C_\omega$ is closed. C is built from Ad_G -invariant functions, but only the first term in (2.11) comes from an invariant polynomial in $\text{Lie}(\mathcal{G})^\mathcal{G}$. Nevertheless, we interpret (2.10) as a sum of “leading order currents” evaluated on ω .

Proposition 2.10. *For $a \in H_2(M^4, \mathbb{Q})$, a representative two-form for Donaldson’s μ -invariant $\mu(a)$ is given by evaluating the leading order two-current*

$$\frac{1}{8\pi^2} \int_M \text{tr}(X \wedge Y) \wedge \cdot + \frac{1}{2\pi^2} \int_M \text{tr}(\Omega_A(X, Y)F_A) \wedge \cdot$$

on any two-form Poincaré dual to a .

As with Gromov-Witten theory, there may be other significant cycles in \mathcal{B}^* not in \mathcal{M} that could be detected by these leading order classes/currents.

Remark 2.1. Except for the gauge theory case, the infinite rank bundles in this section have all been pushdowns of finite rank bundles on the total space of a trivial fibration as in (2.1). In contrast, the Families Index Theorem setup involves a superbundle with superconnection $(E, \nabla) \rightarrow M$ on the total space of a nontrivial fibration of manifolds $Z \rightarrow M \xrightarrow{\pi} B$ with Z spin. It would be very interesting to recast this theorem in terms of the infinite rank pushdown bundle $\pi_* E$. $\pi_* \nabla$ has connection and curvature forms taking values in $\text{End}(E|_{\pi^{-1}(b)})$ for $b \in B$, which is not very exciting. However, the index bundle $\text{IND}(\not{D}_{\nabla^E})$ of the family of twisted Dirac operators on the fibers is a subbundle of $\pi_* E$; with respect to the splitting of $\pi_* E$ into the index bundle and its orthogonal complement, the curvature and connection forms of $\pi_* \nabla$ decompose into matrices with entries locally taking values in zeroth order Ψ DOs in a local trivialization. Thus the group ΨDO_0^* of zeroth order invertible Ψ DOs is related to the Families Index Theorem.

Because the fibration $Z \rightarrow M \rightarrow B$ is nontrivial in general, its structure group is the diffeomorphism group of Z , so it appears that the Ψ DOs glue via Fourier integral operators (FIOs). However, although the fibers $E|_{\pi^{-1}(b)}$ in different trivializations are related by bundle maps covering diffeomorphisms of Z , it seems that the sections of the trivialized fibers are canonically isomorphic. Thus a Ψ DO on a fiber in one trivialization is equivalent to a Ψ DO on the fiber in a different trivialization.

As a result, it is not clear at present if one has to extend the structure group from the group ΨDO_0^* to some larger group of FIOs. For a discussion of invertible FIOs and their Lie algebra, see [14], and for a discussion of perhaps a smaller Lie group, see [20, §6].

3. WODZICKI CLASSES AND APPLICATIONS

In this section we discuss characteristic classes on infinite rank bundles built from the Wodzicki residue, the only trace on the full algebra of Ψ DOs acting on sections of a fixed bundle. We will see that the Pontrjagin or Chern classes of these bundles always vanish, but the associated Wodzicki-Chern-Simons classes can be nonzero. We will then use these WCS classes to study diffeomorphism groups of a class of 5-manifolds.

In particular, we will find several classes of 5-manifolds \overline{M}_k with $\pi_1(\text{Diff}(\overline{M}_k))$ infinite. In general, there seems to be little in the literature about the homotopy type of $\text{Diff}(M)$ once $\dim(M) \geq 3$.

3.1. Wodzicki-Chern-Simons classes. As motivation, we have noted that $T\text{Maps}(N, M)$ is a gauge group bundle, *i.e.*, on the component of a fixed $f \in \text{Maps}(N, M)$, the transition functions lie in the gauge group \mathcal{G} of $f^*TM \rightarrow N$. Thus any \mathcal{G} -connection will have connection one-form and curvature two-form taking values in $\text{Lie}(\mathcal{G})$, an algebra of bundle endomorphisms/multiplication operators. However, the Levi-Civita connections of the natural Riemannian geometry of $\text{Maps}(N, M)$ have connection and curvature forms taking values in a larger group of Ψ DOs. This is similar in spirit to a finite rank hermitian bundle with a non-unitary connection. In the finite rank case, the structure group $GL(n, \mathbb{C})$ deformation retracts onto $U(n)$, so any connection can be unitarized. In our case, the relevant group ΨDO_0^* of invertible zeroth order Ψ DOs acting on sections of *e.g.* $E = f^*TM$ does not retract onto the gauge group.

ΨDO_0^* seems to be an important group in infinite dimensional geometry. It is the intersection of the algebra of all Ψ DOs with the group $GL(\Gamma(E))$ and so is the largest group of Ψ DOs consisting of bounded invertible operators with bounded inverses. The Lie algebra of ΨDO_0^* is $\Psi\text{DO}_{\leq 0}$, the algebra of Ψ DOs of nonpositive order; see [18], where we first learned of the importance of this group and its Lie algebra. Thus we are forced to deal with these Ψ DO-connections directly, and the Wodzicki residue is worth incorporating into Chern-Simons theory.

Recall that a classical Ψ DO P acting on sections of $E \rightarrow M^n$ has an order $\alpha \in \mathbb{R}$ and a symbol expansion $\sigma^P(x, \xi) \sim \sum_{k=0}^{\infty} \sigma_{\alpha-k}^P(x, \xi)$, where $x \in M, \xi \in T_x^*M$, and $\sigma_{\alpha-k}^P(x, \xi)$ is homogeneous of degree $\alpha - k$ in ξ . For (x, ξ) fixed, $\sigma^P(x, \xi), \sigma_{\alpha-k}^P(x, \xi) \in \text{End}(E_x)$. The Wodzicki residue of P is

$$\text{res}^W(P) = \frac{1}{(2\pi)^n} \int_{S^*M} \text{tr} \sigma_{-n}^P(x, \xi) d\xi \, dx, \quad (3.1)$$

where S^*M is the unit cosphere bundle over M with respect to a fixed Riemannian metric. It is nontrivial that res^W is independent of coordinates and defines a trace: $\text{res}^W[P, Q] = 0$. The $\sigma_{\alpha-k}^P$ are computable microlocally at each (x, ξ) , which is crucial for us. In contrast, the equivalent definition $\text{res}^W(P) = \text{res}_{s=0} \text{Tr}(\Delta^{-s}P)$, for any positive order, positive elliptic operator Δ on $\Gamma(E)$,

shows that the Wodzicki residue is a regularized trace, and makes the local expression (3.1) all the more remarkable.

Since the computation complexity of σ_{-n} grows exponentially with n , we will just consider loop spaces ($N = S^1$). As a trace, the Wodzicki residue is an Ad-invariant polynomial on ΨDO_0^* , so we can define Wodzicki-Chern or residue classes for any ΨDO_0^* -connection on LM by

$$c_k^W(TLM) = \frac{1}{k!} \left[\int_{S^*S^1} \text{tr } \sigma_{-1}(\Omega^k) d\xi dx \right] \in H^{2k}(LM, \mathbb{C}). \quad (3.2)$$

These classes always vanish. For $c_k^W(TLM)$ is independent of the connection, and as a gauge bundle, TLM admits a gauge connection whose curvature Ω takes values in multiplication operators, an especially simple subset of $\Psi\text{DO}_{\leq 0}$. The symbol of a multiplication operator is just the operator itself, so $\sigma_{-1}(\Omega^k) = 0$. (It is conjectured that the residue classes vanish for all ΨDO_0^* -bundles.)

Thus we are forced to consider Wodzicki-Chern-Simons (WCS) forms:

Definition 3.1. The k^{th} *Wodzicki-Chern-Simons (WCS) form* of two ΨDO_0^* -connections ∇_0, ∇_1 on TLM is

$$\begin{aligned} CS_{2k-1}^W(\nabla_1, \nabla_0) &= \frac{1}{k!} \int_0^1 \int_{S^*S^1} \text{tr } \sigma_{-1}((\omega_1 - \omega_0) \wedge (\Omega_t)^{k-1}) dt \\ &= \frac{1}{k!} \int_0^1 \text{res}^w[(\omega_1 - \omega_0) \wedge (\Omega_t)^{k-1}] dt. \end{aligned} \quad (3.3)$$

As usual, $dCS_{2k-1}^W(\nabla_1, \nabla_0) = c_k^W(\nabla_0) - c_k^W(\nabla_1)$. Therefore, if $c_k^W(\nabla_0) = c_k^W(\nabla_1) = 0$ pointwise, we get WCS classes $CS_k^W(TLM) \in H^{2k-1}(LM, \mathbb{C})$. Of course, TLM is a real bundle, but unlike in finite dimensions, there is no *a priori* reason for the WCS classes to vanish if k is odd.

Finally, one might wonder if there are traces on ΨDO_0^* besides the leading order trace (and its distributional variants) and the residue trace. In fact, it is a theorem of [10, 11] that there are no more traces. However, analogous to the Pfaffian for $\mathfrak{so}(n)$, there could certainly be Ad-invariant polynomials not built from traces on $\Psi\text{DO}_{\leq 0}$, or on the full algebra of ΨDO s, or on a geometrically interesting subalgebra. One step in this direction is the residue determinant in [21], but a complete theory is unknown at present.

3.2. Levi-Civita connections on LM . If M has a Riemannian metric g , LM has the L^2 metric (2.5), which was important for the S^1 -index theorem discussion in §2. On its own, this metric is not so interesting: its curvature $\Omega(X, Y)_\gamma(\theta) = \Omega^M(X(\theta), Y(\theta))_{\gamma(\theta)}$ contains no more information than the curvature of M . It is much more fruitful to pick a parameter $s \gg 0$ and define the

s -Sobolev or H^s -metric by

$$\langle X, Y \rangle_s = \frac{1}{2\pi} \int_0^{2\pi} \langle (1 + \Delta)^s X(\alpha), Y(\alpha) \rangle_{\gamma(\alpha)} d\alpha, \quad X, Y \in \Gamma(\gamma^* TM). \quad (3.4)$$

Here $\Delta = D^*D$, with $D = D/d\gamma$ the covariant derivative along γ . For $s \in \mathbb{Z}^+$, $(1 + \Delta)^s$ is a differential operator, while for nonintegral s , it is a Ψ DO of order $2s$. (Here TLM is modeled on $H^{s'}$ sections of $\gamma^* TM$ with $s' \gg s$.) The use of $(1 + \Delta)^s$ is a standard analytic trick to impose regularity: X is at least $s - 1$ times differentiable if $\|X\|_s < \infty$. Note that $s = 0$ recovers the L^2 metric. From a physics point of view, we think of s as a parameter we would like to set equal to infinity. Since that is impossible, we want to extract information from these metrics that is independent of s .

It was shown in [5] that the Levi-Civita connection for the H^s metric on loop groups has connection one-form taking values in Ψ DOs. In [12], this is extended to general loop spaces. We only state the result for $s = 1$.

Theorem 3.1. *Let ∇^0 be the Levi-Civita connection for the L^2 metric on LM , let ∇^M be the Levi-Civita connection on M , and let Ω^M be its curvature two-form. The $s = 1$ Levi-Civita connection ∇^1 on LM is given at the loop γ by*

$$\begin{aligned} \nabla_X^1 Y = & \nabla_X^0 Y + \frac{1}{2}(1 + \Delta)^{-1} \left[-\nabla_{\dot{\gamma}}^M (\Omega^M(X, \dot{\gamma})Y) - \Omega^M(X, \dot{\gamma}) \nabla_{\dot{\gamma}}^M Y - \nabla_{\dot{\gamma}}^M (\Omega^M(Y, \dot{\gamma})X) \right. \\ & \left. - \Omega^M(Y, \dot{\gamma}) \nabla_{\dot{\gamma}}^M X + \Omega^M(X, \nabla_{\dot{\gamma}}^M Y) \dot{\gamma} - \Omega^M(\nabla_{\dot{\gamma}}^M X, Y) \dot{\gamma} \right]. \end{aligned} \quad (3.5)$$

This is proven by examining the six-term formula as in (2.6). As an operator on Y , $\nabla_X^0 Y$ is zeroth order, while all other terms are order -1 or -2 . For example, in the term $(1 + \Delta)^{-1} \nabla_{\dot{\gamma}}^M (\Omega^M(X, \dot{\gamma})Y)$, $(1 + \Delta)^{-1}$ has order -2 and $\nabla_{\dot{\gamma}}^M (\Omega^M(X, \dot{\gamma})Y)$ contains subterms of order 0 and 1 in Y . Since orders add under composition of operators, the subterms have orders as stated. Although the appearance of the covariant derivative of the curvature in (3.5) is unwelcome, the Levi-Civita connection is explicit, so that the symbol asymptotics of the curvature Ω^1 of ∇^1 can be computed to any order [12, Appendix]. Not surprisingly, Ω^1 equals Ω^0 plus a Ψ DO of order at most -1 .

This fits very well with Def. 3.1 with ∇_0, ∇_1 the L^2 and $s = 1$ Levi-Civita connections. $\omega_1 - \omega_0$ has strictly negative order, while Ω_t has its order 0 term given by classical curvature expressions. This makes the integrand $\text{res}^w[(\omega_1 - \omega_0) \wedge (\Omega_t)^{k-1}]$ in (3.3) relatively straightforward to compute.

Theorem 3.2. *Let $\dim(M) = 2k - 1$. Fix a Riemannian metric on M with curvature two-form Ω^M , and fix $X_1, \dots, X_{2k-1} \in T_\gamma LM$. The k^{th} Wodzicki-Chern-Simons form $CS_{2k-1}^W(\nabla^1, \nabla^0)$ is*

given by

$$\begin{aligned} CS_{2k-1}^W(\nabla^1, \nabla^0)(X_1, \dots, X_{2k-1}) \\ = \frac{4}{(2k-1)!} \sum_{\sigma} \text{sgn}(\sigma) \int_{S^1} \text{tr}[(\Omega^M(X_{\sigma(1)}, \cdot) \dot{\gamma})(\Omega^M)^{k-1}(X_{\sigma(2)}, \dots, X_{\sigma(2k-1)})]. \end{aligned} \quad (3.6)$$

Here σ is a permutation of $\{1, \dots, 2k-1\}$.

In contrast, WCS classes for string theory, *i.e.*, $\text{Maps}(\Sigma^2, M)$, are harder to compute, since $T\text{Maps}(\Sigma^2, M)$ is modeled on sections of a bundle over Σ^2 . Although the connection and curvature forms still take values in zeroth order Ψ DOs, the Wodzicki residue now involves σ_{-2} , which means that covariant derivatives of Ω^M appear in the analog of Thm. 3.2.

Remark 3.1. These two theorems indicate that other traces on negative order Ψ DOs, classified in [10], do not appear naturally in this context. The zeroth order part of the connection and curvature forms come from the corresponding forms on M , so the connection and curvature forms of the H^s metric on LM take values in negative order Ψ DOs only if M is flat. Even though LM is nontrivial in this case, the WCS forms will vanish. The same remarks hold in string theory with flat target manifolds.

For degree reasons, the form $c_k^W(\Omega) = (k!)^{-1} \int_{S^*S^1} \text{tr} \sigma_{-1}(\Omega^k)$ vanishes for $\dim(M) = 2k-1$. Thus the WCS class

$$[CS_{2k-1}^W(\nabla^1, \nabla^0)] \in H^{2k-1}(LM, \mathbb{C}) \quad (3.7)$$

is defined.

If we use the H^s Levi-Civita connection, we obtain $CS_{2k-1}^W(\nabla^s, \nabla^0) = s \cdot CS_{2k-1}^W(\nabla^1, \nabla^0)$. Therefore the s -independent information in this WCS class is given by setting $s = 1$; in physics terminology, we have successfully regularized the WCS class.

In contrast to finite dimensions, CS_3^W vanishes pointwise on 3-manifolds due to symmetries of the curvature tensor. Thus we will consider 5-manifolds.

3.3. WCS classes and diffeomorphism groups. In this subsection, we produce several infinite families of 5-manifolds \overline{M}_k with $|\pi_1(\text{Diff}(\overline{M}_k))| = \infty$.

In general, information about $\text{Diff}(M)$ seems very difficult to come by. For example, it is a theorem of Smale that $\text{Diff}(S^2) \sim O(3)$, where the tilde means homotopy equivalence, and a theorem of Hatcher that $\text{Diff}(S^3) \sim O(4)$. There is a good understanding of the homotopy type of the identity component of $\text{Diff}(M^2)$ and $\text{Diff}(M^3)$ for M^3 hyperbolic or Seifert fibered. In addition, one knows the stable homotopy groups of $\text{Diff}(S^n)$ modulo torsion. These are all difficult results, and use very different techniques from ours.

Our main result Thm. 3.6 states that for every projective algebraic Kähler surface M , there is an infinite family \overline{M}_k of 5-manifolds with $\pi_1(\text{Diff}(\overline{M}_k))$ infinite. For specific Kähler surfaces, we can give more precise information.

To begin the construction of \overline{M}_k , recall that an S^1 action $a : S^1 \times M \rightarrow M$ induces $a^L : M \rightarrow LM$, $a^D : S^1 \rightarrow \text{Diff}(M)$ by $a^L(m)(\theta) = a^D(\theta)(m) = a(\theta, m)$. Clearly a^L and a^D are closely related, and the following Lemma makes this explicit. For notation, let $[a^L]$ denote $a_*^L[M] \in H_{\dim(M)}(M, \mathbb{C})$.

Lemma 3.3. *Let $\dim(M) = 2k - 1$, and let $a_0, a_1 : S^1 \times M \rightarrow M$ be actions.*

(i) *Let α be a closed form on LM of degree $2k - 1$. If $\int_{[a_0^L]} \alpha \neq \int_{[a_1^L]} \alpha$, then $[a_0^D] \neq [a_1^D] \in \pi_1(\text{Diff}(M), \text{Id})$.*

(ii) *If $\int_{[a_1^L]} CS_{2k-1}^W \neq 0$, then $\pi_1(\text{Diff}(M), \text{Id})$ is infinite.*

Here and from now on, CS_{2k-1}^W denotes $CS_{2k-1}^W(\nabla^1, \nabla^0)$.

Sketch of proof. (i) By Stokes' theorem, $[a_0] \neq [a_1^L] \in H_{2k-1}(LM, \mathbb{C})$. It follows that a_0 and a_1 are not homotopic, which implies that $[a_0^D] \neq [a_1^D]$. See [13] for details.

(ii) Let a_n be the n^{th} iterate of a_1 : $a_n(\theta, m) = a_1(n\theta, m)$. We outline a proof that $\int_{[a_n^L]} CS_{2k-1}^W = n \int_{[a_1^L]} CS_{2k-1}^W$. For by (3.6), every term in CS_{2k-1}^W is of the form $\int_{S^1} \dot{\gamma}(\theta) f(\theta)$, where f is a periodic function on the circle. Each loop $\gamma \in a_1^L(M)$ corresponds to the loop $\gamma(n \cdot) \in a_n^L(M)$. Therefore $\int_{S^1} \dot{\gamma}(\theta) f(\theta)$ is replaced by

$$\int_{S^1} \frac{d}{d\theta} \gamma(n\theta) f(n\theta) d\theta = n \int_0^{2\pi} \dot{\gamma}(\theta) f(\theta) d\theta.$$

Thus $\int_{[a_n^L]} CS_{2k-1}^W = n \int_{[a_1^L]} CS_{2k-1}^W$. By (i), the $[a_n^L] \in \pi_1(\text{Diff}(M), \text{Id})$ are all distinct. \square

Lemma 3.3(ii) gives us a strategy to produce 5-manifolds M with infinite $\pi_1(\text{Diff}(M))$. We want an S^1 action a and a relatively computable metric on M . If $\int_{a_*[M]} CS_5^W \neq 0$, then $|\pi_1(\text{Diff}(M))| = \infty$. From examples in the literature, especially [6], it seems best to consider the total space of a circle bundle over a Kähler surface, as these spaces have an obvious S^1 action by rotating the circle fibers and carry Sasakian metrics closely related to the Kähler metric.

As pointed out to us by Alan Hatcher, it is not always the case that the fiber rotation is an element of infinite order in $\pi_1(\text{Diff}(M))$. For the free action of S^1 on $S^5 \subset \mathbb{C}^3$ given by $a(e^{i\theta}, z) = e^{i\theta} z$ has quotient $M = \mathbb{CP}^2$. The action is via isometries for the standard metric on S^5 , and so gives an element in $\pi_1(\text{Isom}(S^5)) = \pi_1(SO(6)) = \mathbb{Z}_2$. Under the inclusion $\text{Isom}(S^5) \rightarrow \text{Diff}(S^5)$, this element has order at most two.

In general, let (M^4, g, J, ω) be an integral Kähler surface, i.e. J is the complex structure, g is the Kähler metric, and the Kähler form is $\omega \in H^2(M, \mathbb{Z})$. It follows from the Kodaira embedding theorem that M is integral iff it is projective algebraic. Fix $k \in \mathbb{Z}$. As in geometric quantization, we can construct a S^1 -bundle $L_k \rightarrow M$ with connection $\bar{\eta}$ with curvature $d\bar{\eta} = k\omega$. Let \bar{M}_k be the total space of L_k .

\bar{M}_k has a Sasakian structure; see [3, §4.5], [13], [17, Lemma 1] for details. The horizontal space of the connection is $\mathcal{H} = \text{Ker}(\bar{\eta})$. Define a metric \bar{g} on \bar{M}_k by

$$\bar{g}(\bar{X}, \bar{Y}) = g(\pi_* \bar{X}, \pi_* \bar{Y}) + \bar{\eta}(\bar{X})\bar{\eta}(\bar{Y}).$$

Let R, \bar{R} be the curvature tensors for g, \bar{g} , respectively. By some careful computations relying heavily on the fact that g is Kähler, we obtain:

Lemma 3.4.

$$\begin{aligned} \bar{g}(\bar{R}(X^L, Y^L)Z^L, W^L) &= \langle R(X, Y)Z, W \rangle + k^2[-\langle JY, Z \rangle \langle JX, W \rangle \\ &\quad + \langle JX, Z \rangle \langle JY, W \rangle + 2\langle JX, Y \rangle \langle JZ, W \rangle], \\ \bar{g}(\bar{R}(X^L, Y^L)Z^L, \bar{\xi}) &= 0, \\ \bar{g}(\bar{R}(\bar{\xi}, X^L)Y^L, \bar{\xi}) &= k^2 \langle X, Y \rangle. \end{aligned}$$

We want to show

$$0 \neq \int_{[a^L]} CS_5^W = \int_{a_*[\bar{M}_k]} CS_5^W = \int_{\bar{M}_k} a^* CS_5^W.$$

$a^* CS_5^W$ is a multiple f of the volume form on \bar{M}_k . If $\bar{\xi}$ is a unit length vertical vector and (e_2, Je_2, e_3, Je_3) is a positively oriented orthonormal frame on M , then $f = CS_{5,\gamma}^W(\bar{\xi}, e_2, Je_2, e_3, Je_3)$. A long computation in [13] using (3.6) and the previous Lemma gives

$$\begin{aligned} &CS_{5,\gamma}^W(\bar{\xi}, e_2, Je_2, e_3, Je_3) \\ &= \frac{k^2}{30} \{ 32\pi^2 p_1(R)(e_2, Je_2, e_3, Je_3) + 32k^2 [3R(e_2, Je_2, e_3, Je_3) - R(e_2, e_3, e_2, e_3) \\ &\quad - R(e_2, Je_3, e_2, Je_3) + R(e_2, Je_2, e_2, Je_2) + R(e_3, Je_3, e_3, Je_3)] \\ &\quad + 192k^4 \}, \end{aligned} \tag{3.8}$$

where $p_1(R)$ is the first Pontrjagin form. This leads to a crucial estimate. Set

$$|R|_\infty = \max_E \{|R(e_i, e_j, e_k, e_\ell)|\},$$

where E is the set of orthonormal frames at all points of M .

Proposition 3.5. $\int_{\overline{M}_k} CS_5^W > 0$ if

$$k^2 (96\pi^2 \sigma(M) - 224k^2 |R|_\infty \text{vol}(M) + 192k^4 \cdot \text{vol}(M)) > 0.$$

Here $\sigma(M) = \frac{1}{3} \int_M p_1(R)$ is the signature of M . Since the k^4 term will dominate for $k \gg 0$, we get

Theorem 3.6. *Let (M^4, J, g, ω) be a compact integral Kähler surface, and let \overline{M}_k be the circle bundle associated to $k[\omega] \in H^2(M, \mathbb{Z})$ for $k \in \mathbb{Z}$. Then the loop of diffeomorphisms of \overline{M}_k given by rotation in the circle fiber gives an element of infinite order in $\pi_1(\text{Diff}(\overline{M}_k))$ for $|k| \gg 0$. This loop is also an element of infinite order in $\pi_1(\text{Isom}(M))$.*

The last statement follows as in the S^5 example.

We note that these results tell us nothing if $k = 0$, i.e. for $\overline{M}_0 = M \times S^1$. One would think this is the easiest case, but our methods fail here.

For specific Kähler metrics, we can give more precise results using (3.8). For notation, on $M = \mathbb{CP}^1 \times \mathbb{CP}^1$, let ω_1, ω_2 be the standard Kähler form on each \mathbb{CP}^1 with sectional curvature 1. For $a, b \in \mathbb{Z}^+$, let $\omega = a\omega_1 + b\omega_2$ be an integral Kähler form on M , and let $\overline{M}_{k(a,b)}$ be the total space of the line bundle associated to $k\omega$. For M a projective algebraic K3 surface, recall that $H^2(M) \simeq \mathbb{Z}^{22}$. Fix an integral Kähler class $[\omega] = [\omega_1, \dots, \omega_{22}]$ in the obvious notation. Take $a_1, \dots, a_{22} \in \mathbb{Z}^+$. For $k \in \mathbb{Z} \setminus \{0\}$, let $\overline{M}_{k\vec{a}}$ be the total space of the line bundle associated to $k \sum_{i=1}^{22} a_i \omega_i$.

Theorem 3.7. (i) $\pi_1(\text{Diff}(\overline{T}_k^4))$ is infinite for $k \neq 0$.

(ii) $\pi_1(\text{Diff}(\overline{\mathbb{CP}}_k^2))$ is infinite for $k \neq 0, \pm 1$.

(iii) For $a, b \in \mathbb{Z}^+, k \neq 0$, $\pi_1(\text{Diff}(\overline{M}_{k(a,b)}))$ is infinite.

(iv) Let M be a projective algebraic K3 surface. $\pi_1(\text{Diff}(\overline{M}_{k\vec{a}}))$ is infinite for $k \neq 0$.

(v) There are infinitely many values of $k_1, k_2, k_3, k_4, a, b, \vec{a}$ such that $\overline{T}_{k_1}^4, \overline{\mathbb{CP}}_{k_2}^2, \overline{M}_{k_3(a,b)}, \overline{M}_{k_4\vec{a}}$ are mutually nonhomeomorphic.

(i) follows immediately from Prop. 3.5, since $|R|_\infty = 0$ on the flat torus and $\sigma(T^4) = 0$. For (ii), (3.8) vanishes only for $k = 0, \pm 1$, as it must; this gives us confidence that the constants in (3.8) are correct. (iii) uses the Ricci flat metric on a K3 surface and the decomposition of $\Lambda^2(M)$ into selfdual and anti-selfdual forms. (iv) follows from Gysin sequence computations of the cohomology of these spaces. Details are in [13].

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